How to trade large block of shares?

Introduction to optimal execution strategies

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Introduction

- We consider the problem of unwinding a single-stock portfolio with position $q_0 > 0$ over the time interval $[0, T]$.

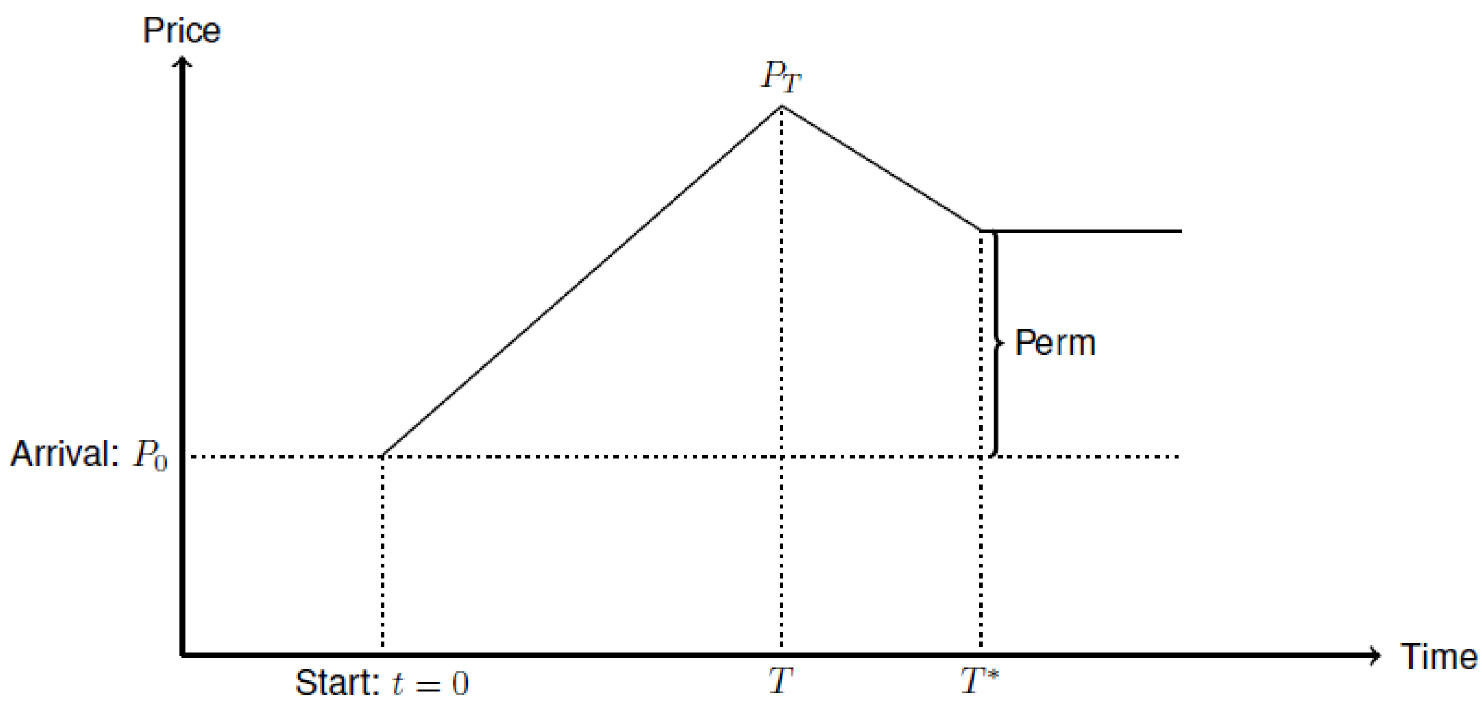
- The trader's position is modeled by the process $(q_t)_{t\in[0,T]}$ with the dynamics

$$dq_t = v_t dt$$

where $\mathbf{v} = (v_t)_{t\in[0,T]}$ satisfies the unwinding constraint $\int_0^T v_t dt = -q_0$.

Permanent vs instantaneous (temporary) market impact

It is customary to decompose market impact costs of a trade into its permanent and temporary constituent parts



Normal dynamics vs. log-normal dynamics

We denote by $\mathcal{A}$ the set of admissible controls **v**.

The mid-price of the stock is modeled by the process $(S_t)_{t\in[0,T]}$, where $S_t = S_t^{\text{mid}} = \frac{1}{2} \cdot \left(S_t^{\text{bid}} + S_t^{\text{ask}}\right)$

$$dS_t = \sigma dW_t + kv_t dt$$

σ - the arithmetic volatility

$k \geq 0$ - the magnitude of the permanent market impact

Execution costs & cash account process

• Let introduce the market volume process $(V_t)_{t\in[0,T]}$, which represents the velocity of the volume traded by other agents (it does not take into account our own trades). We assume that it is a deterministic, continuous, positive, and bounded process.

• The price obtained by the trader for each share at time t is of the form $S_t + g\left(\frac{v_t}{V_t}\right)$, where g is an increasing function satisfying $g(0) = 0$. It models the instantaneous market impact.

• $L(\rho) = \rho g(\rho)$ - execution cost function

• We denote by $(X_t)_t$ the cash account process modeling the amount of cash on the trader's account. It's dynamics is given by

$$dX_t = -v_t\left(S_t + g\left(\frac{v_t}{V_t}\right)\right)dt = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right)dt$$

Assumptions on the function $L\colon \mathbb{R} \to \mathbb{R}$

The assumptions on the function $L\colon \mathbb{R} \to \mathbb{R}$ are the following:

- No fixed cost, i.e., $L(0) = 0$,
- L is strictly convex, increasing on $\mathbb{R}_+$ and decreasing on $\mathbb{R}_-$,
- L is asymptotically super-linear, i.e., $\lim_{|\rho|\to\infty} \frac{L(\rho)}{|\rho|} = +\infty$.

In practical examples

$$L(\rho) = |\rho|^{1+\phi} \qquad \text{or} \qquad L(\rho) = |\rho|^{1+\phi} + \psi|\rho|$$

where the additional term $\psi|\rho|$ models proportional costs such as the bid-ask spread.

The initial A-Ch models correspond to a quadratic function $L(\rho) = \eta\rho^2$.

Why should permanent market impact be linear?

Let assume that the permanent market impact is modeled by the function $\mathcal{I}(\cdot)$. The dynamics of (q_t, S_t, X_t) is

$$\begin{cases} dq_t = & v_t dt \\ dS_t = & \sigma dW_t + \mathcal{I}(v_t) dt \\ dX_t = & -v_t S_t dt \end{cases}$$

There is a dynamic arbitrage if there exist $t_1 < t_2$, and a process $(v_t)_t$ such that the following conditions are satisfied:

- $\int_{t_1}^{t_2} v_t dt = 0$,
- $\mathbb{E}[X_{t_2} | \mathcal{F}_{t_1}] > X_{t_1}$.

In other words, a dynamic arbitrage corresponds to a round trip strategy on the stock that is profitable on average.

Linear permanent market impact guarantees no dynamic arbitrage.

• There is no dynamic arbitrage iff $\mathcal{I}(\cdot)$ is a linear function.

$$\text{We choose } \mathcal{I}(v) = kv \text{ with } k \geq 0.$$

• The permanent component of market impact can also depend on the number of shares already traded. For instance, a model where the price dynamics is

$$dS_t = \sigma dW_t + f\left(|q_0 - q_t|\right) v_t dt$$

with f a positive function (usually decreasing), does not lead to any dynamic arbitrage.

Optimization problem

- Our goal is to find an optimal strategy $(v_t)_t \in \mathcal{A}$ to liquidate the portfolio.

- Mean-variance criterion: maximize $\mathbb{E}[X_T] - \frac{\gamma}{2}\mathbb{V}[X_T]$

- We consider an expected utility criterion. The utility function we consider is a CARA (Constant Absolute Risk Aversion) utility function, that is, an exponential utility function.

- Our objective function is of the form $\mathbb{E}[-\exp(-\gamma X_T)]$

- γ - absolute risk aversion coefficient

Deterministic strategies

- We restrict liquidation strategies to deterministic ones $\mathcal{A}_{\text{det}} \subset \mathcal{A}$.

$$dS_t = \sigma dW_t + kv_t dt$$

$$dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt$$

- The final value of the cash account process is given by

$$\begin{aligned} X_T &= X_0 - \int_0^T v_t S_t dt - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt \\ &= X_0 + q_0 S_0 + \int_0^T k v_t q_t dt + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt \\ &= X_0 + q_0 S_0 - \frac{k}{2} q_0^2 + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt \end{aligned}$$

Deterministic strategies

$$X_T = X_0 + q_0 S_0 - \frac{k}{2} q_0^2 + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt$$

If $(v_t)_{t\in[0,T]} \in \mathcal{A}_{\mathrm{det}}$, then X_T is normally distributed with mean

$$\mathbb{E}\left[X_T\right] = X_0 + q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt,$$

and variance

$$\mathbb{V}\left[X_T\right] = \sigma^2 \int_0^T q_t^2 dt.$$

The mean of X_T can be decomposed into three parts:

$$\mathbb{E}\left[X_T\right] = \underbrace{X_0 + q_0 S_0}_{\text{MtM value}} - \underbrace{\frac{k}{2} q_0^2}_{\text{perm. m. i.}} - \underbrace{\int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt}_{\text{execution costs}}$$

Moment-generating function - two sided Laplace transform of density

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $\gamma > 0$. Then

$$\mathbb{E}\left[-\exp\left(-\gamma X\right)\right] = -\exp\left(-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right)$$

Using moment-generating function of a Gaussian variable, we can compute the value of the objective function:

$$\begin{aligned}
\mathbb{E}\left[-\exp\left(-\gamma X_T\right)\right] &= -\exp\left(-\gamma\mathbb{E}\left[X_T\right] + \frac{1}{2}\gamma^2\mathbb{V}\left[X_T\right]\right) \\
&= -\exp\left(-\gamma\left(X_0 + q_0 S_0 - \frac{k}{2}q_0^2\right)\right) \\
&\times \exp\left(\gamma\left(\int_0^T V_t L\left(\frac{v_t}{V_t}\right)dt + \frac{\gamma}{2}\sigma^2\int_0^T q_t^2 dt\right)\right).
\end{aligned}$$

Optimisation problem

As a consequence the problem boils down to finding a control process $(v_t)_{t\in[0,T]} \in \mathcal{A}_{\text{det}}$ minimizing

$$\int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt + \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt.$$

Because $v_t = \frac{dq_t}{dt}$, the problem boils down to a variational problem (Bolza problem). We need to find minimizers of the functional J defined by

$$J(q) = \int_0^T \left(V_t L\left(\frac{q^{'}(t)}{V_t}\right) + \frac{1}{2}\gamma\sigma^2 q(t)^2 \right) dt$$

over the set of absolutely continuous functions q satisfying the constraints $q(0) = q_0$ and $q(T) = 0$. (There exists a unique minimizer q^*-nonincreasing.)

Characterisation of the optimal strategy

To characterize the optimal strategy q^* we can use an Euler-Lagrange characterization.

$$F(q) = \int_a^b f\left(t, q(t), q^{'}(t)\right) dt, \qquad f(\cdot) = f(t, x, v)$$

$$f_x\left(t, q(t), q^{'}(t)\right) = \frac{d}{dt} f_v\left(t, q(t), q^{'}(t)\right)$$

If L is differentiable then the Euler-Lagrange equation reduces to

$$\begin{cases} p'(t) = & \gamma\sigma^2 q^*(t) \\ p(t) = & L'\left(\frac{{q^*}'(t)}{V_t}\right) \\ q^*(0) = & q_0 \\ q^*(T) = & 0. \end{cases}$$

Legendre Fenchel transform

Let H be the Legendre-Fenchel transform of the function L defined by

$$H(p) = \sup_{\rho} \left(\rho p - L(\rho) \right).$$

Because L is strictly convex, H is a function of class C^1.

We know that $p(t) = L' \left(\frac{q^{*'}(t)}{V_t} \right)$ and the L-F transform can be specified by the condition

$$H' = \left(L' \right)^{-1}$$

Then we get a characterization of the E-L system by the Hamiltonian system:

$$\begin{cases} p'(t) = & \gamma \sigma^2 q^*(t) \\ q^{*'}(t) = & V(t) H'(p(t)) \\ q^*(0) = & q_0 \\ q^*(T) = & 0 \end{cases}$$

The case of quadratic execution costs

In general the solution is given by: $\quad p''(t) = \gamma\sigma^2 V_t H'(p(t))$

We can prove that in our model the optimal deterministic strategy is the best in class of all (deterministic/stochastic) admissible controls. (see *Guéant*'s book)

Let us take $L(\rho) = \eta\rho^2$. Using the characterisation $H' = (L')^{-1}$, the associated Hamiltonian function is $H(p) = \frac{p^2}{4\eta}$.

$$\begin{cases} p'(t) = & \gamma\sigma^2 q^*(t), \\ q^{*'}(t) = & V_t H'(p) = \frac{V_t}{2\eta} p(t), \\ q^*(0) = & q_0, \\ q^*(T) = & 0. \end{cases}$$

Consequently, q^* is the unique solution of the equation

$$q^{*''}(t) = \frac{\gamma\sigma^2 V_t}{2\eta} q^*(t),$$

satisfying the boundary conditions $q^*(0) = q_0$ and $q^*(T) = 0$.

Classical Almgren-Chriss formula

If $(V_t)_t$ is assumed to be constant (i.e. $V_t = V, \forall t \in [0, T]$), then we get the classical hyperbolic sine formula of Almgren and Chriss:

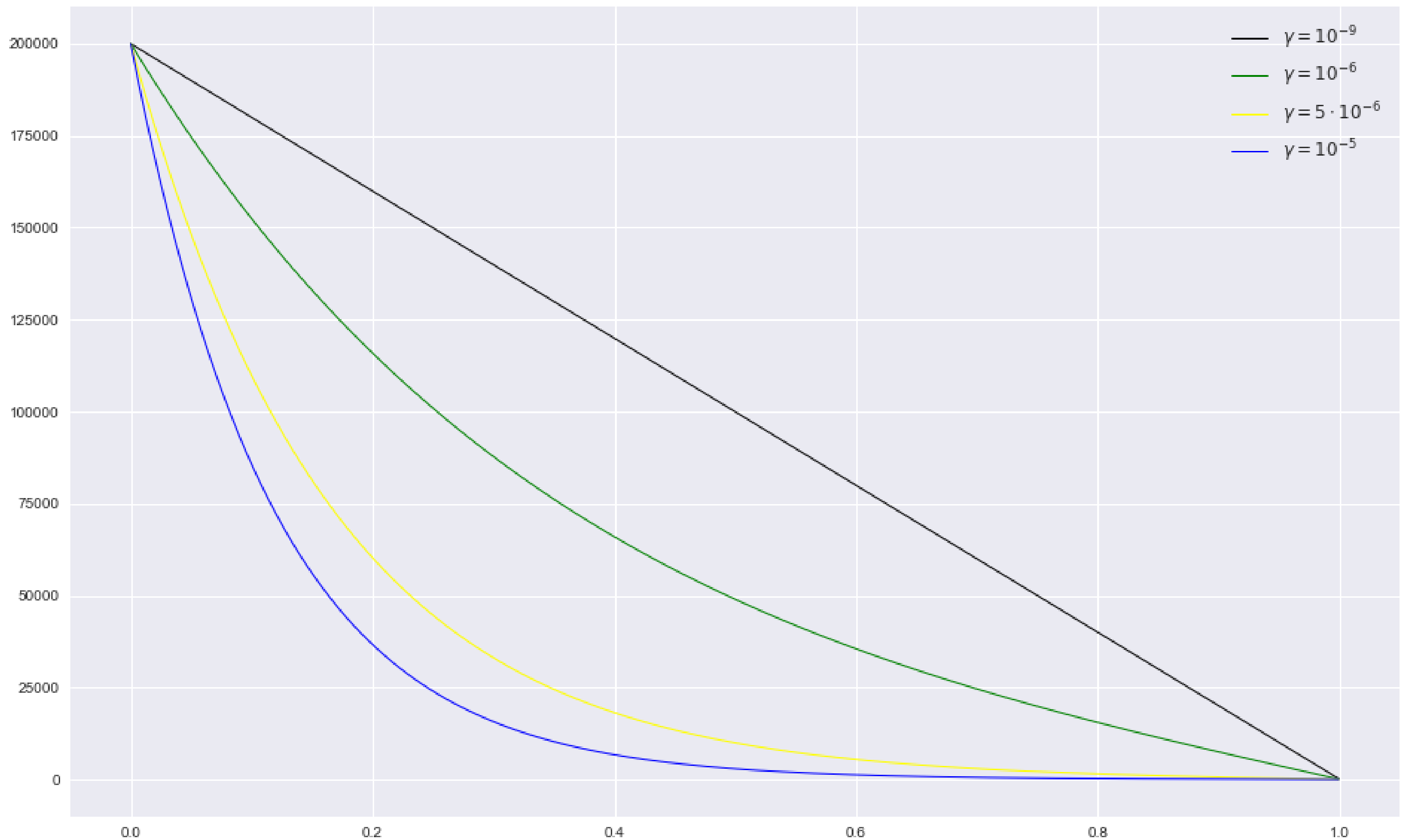
$$q^*(t) = q_0 \frac{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}$$

Associated to this optimal trading curve, the optimal (deterministic) strategy $(v_t^*)_t$ is given by

$$v_t^* = q^{*'}(t) = -q_0\sqrt{\frac{\gamma\sigma^2 V}{2\eta}} \frac{\cosh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}.$$

Optimal trading curves for different γ parameters

$S_0 = 45$ €$, \sigma = 0.6 \cdot \text{day}^{-\frac{1}{2}} \cdot \text{share}^{-1}$(annual volatility 21%)$, q_0 = 200,000$ shares
$V = 4,000,000$ shares $\cdot\, \text{day}^{-1}, \eta = 0.1$ € $\cdot\, \text{share}^{-1}, T = 1$ (day)



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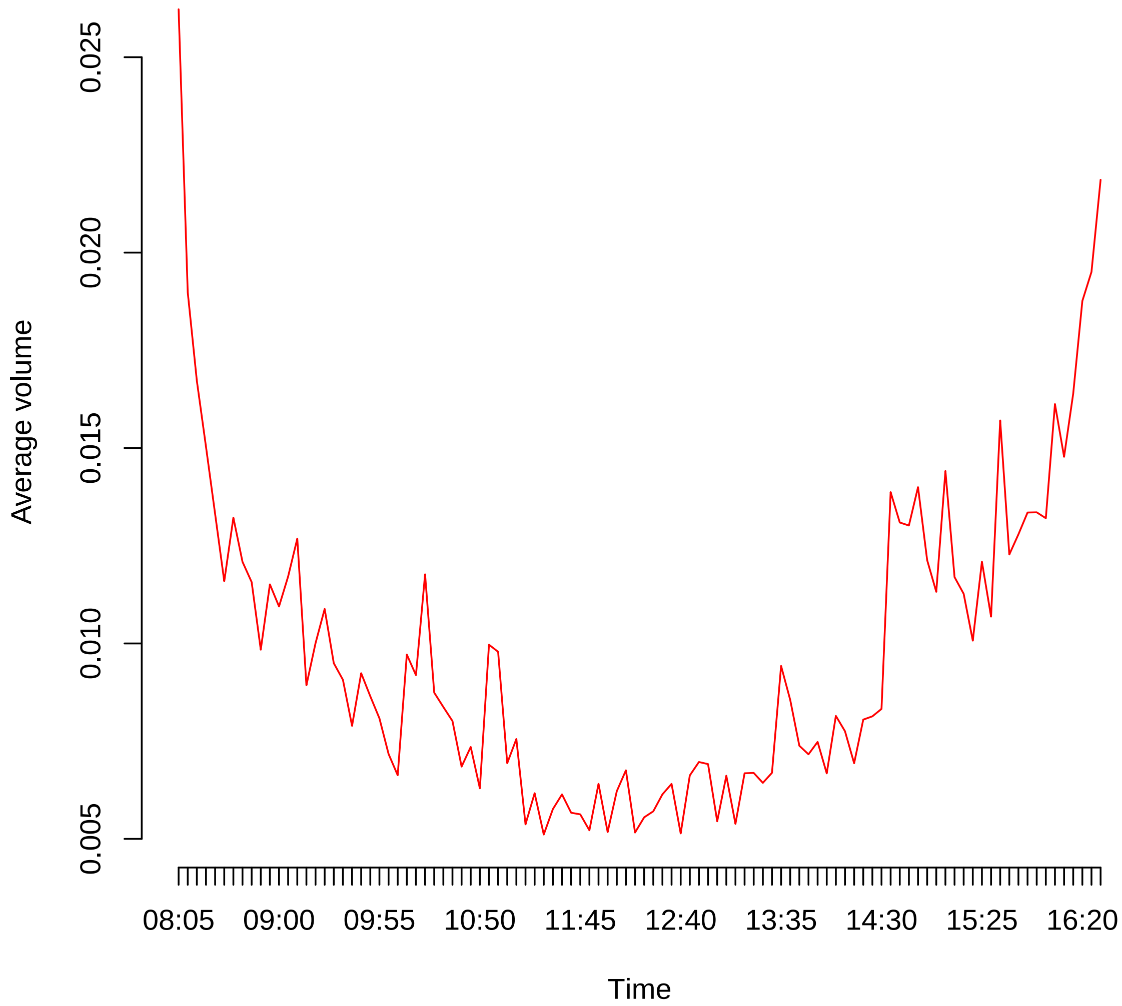
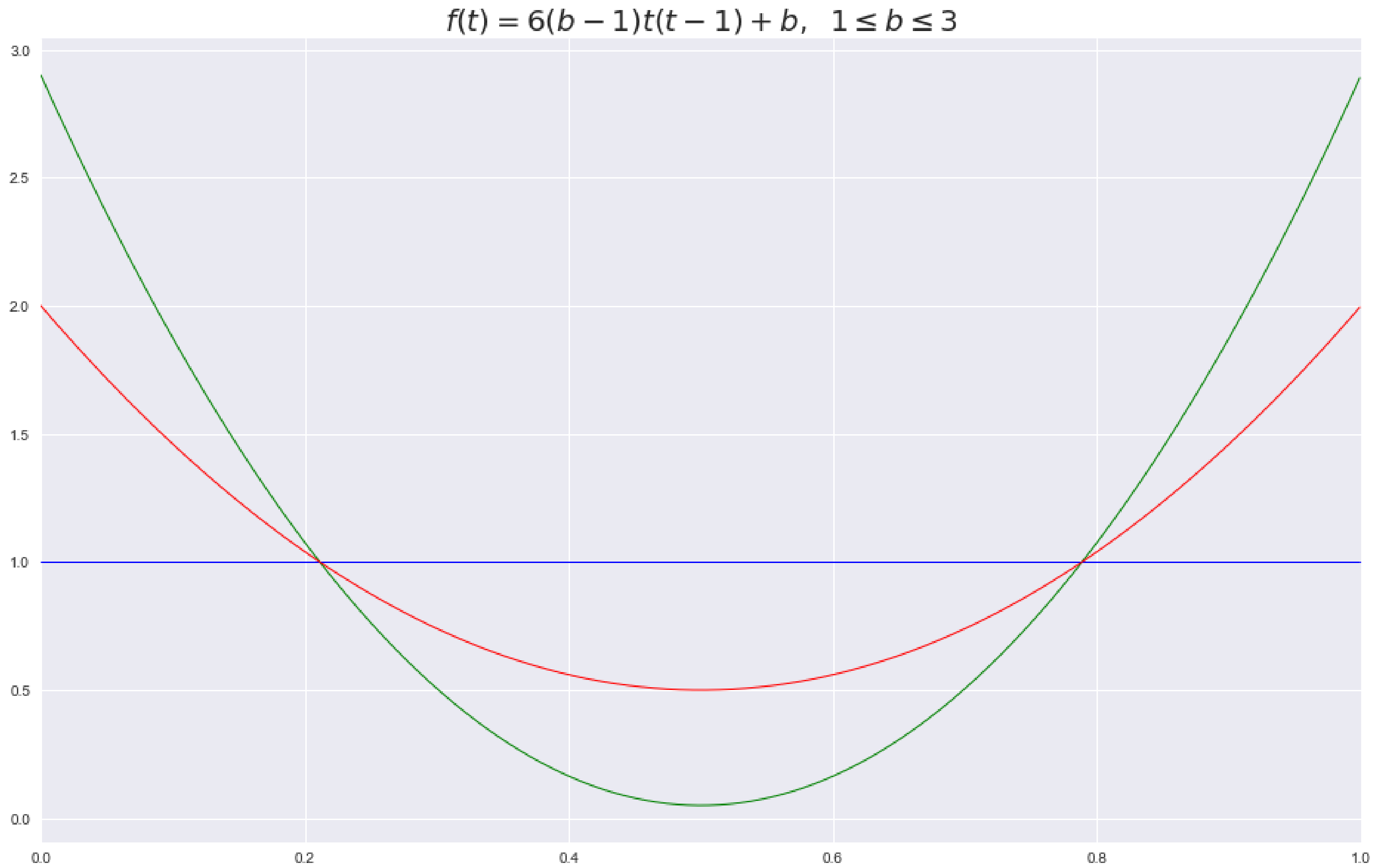
Normalised average volume for BNP Paribas aggregated in 5 min bins



Constant vs quadratic $V_t = V \cdot f(t)$



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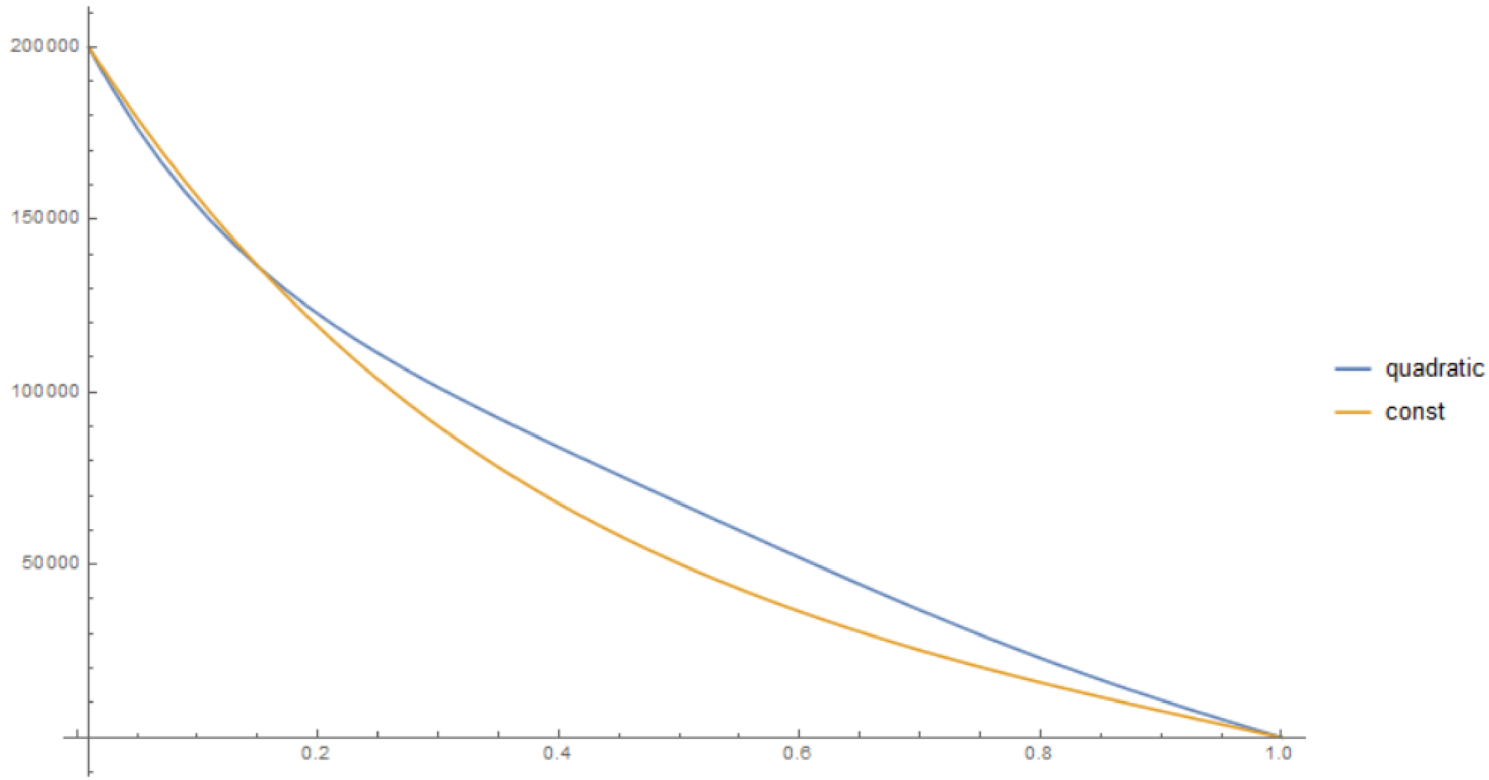
Constant vs quadratic V_t

$q_0 = 200,000$, $V = 4,000,000, \sigma = 0.6, \eta = 0.1, \gamma = 10^{-6}$, $b = 2.9$, $T = 1$



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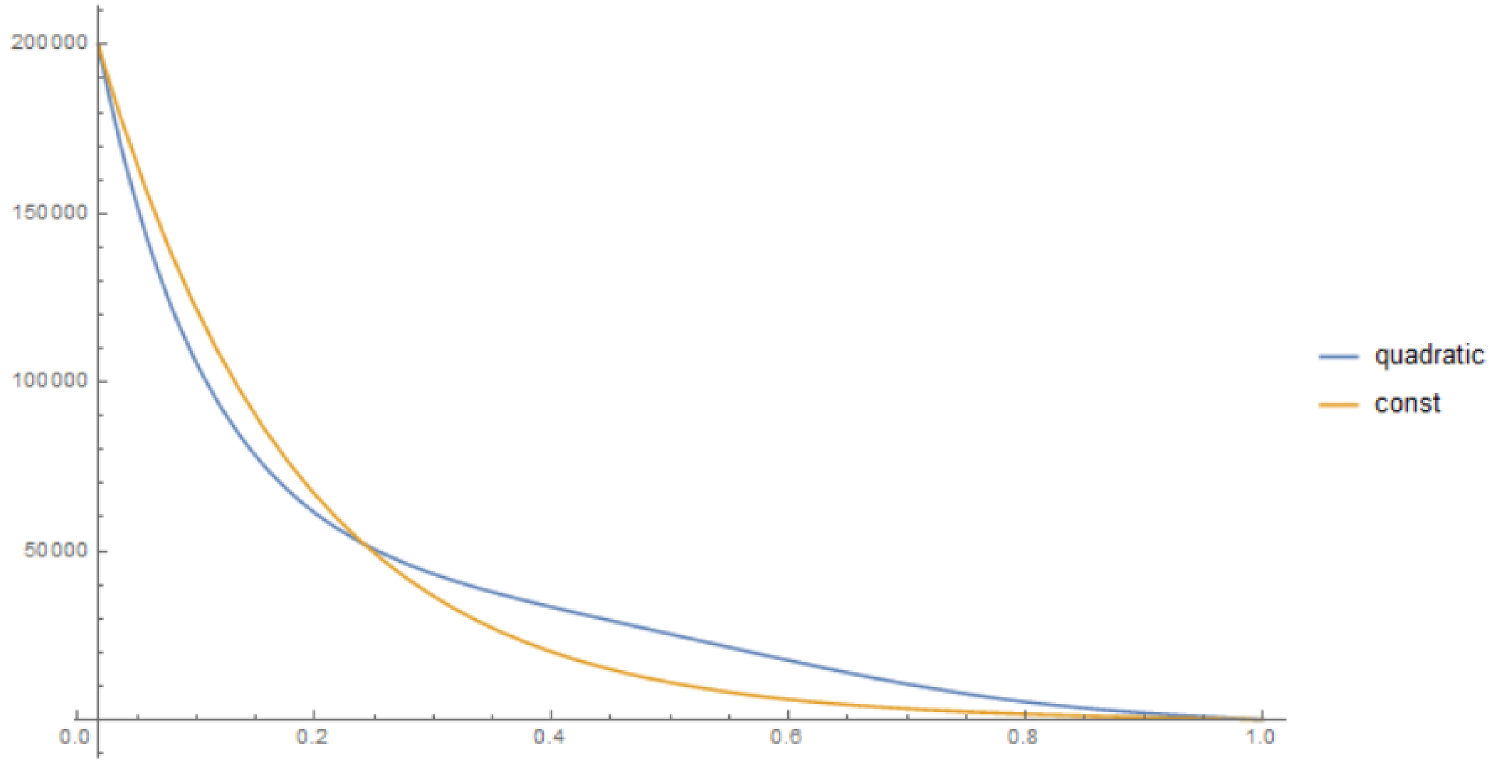
Constant vs quadratic V_t

$q_0 = 200,000$, $V = 40,000,000$, $\sigma = 0.6$, $\eta = 0.1$, $\gamma = 10^{-6}$, $b = 2.999$, $T = 1$



Extensions of the Almgren-Chriss framework

- Optimisation of different benchmark orders, e.g. optimal participation rate for POV, target close algo,
- Different dynamics, e.g. $dS_t = \mu_t dt + \sigma_t dW_t + kv_t dt$ with μ_t, σ_t-deterministic,
- Model with constraints, e.g. minimal participation rate,
- Portfolio liquidation,
- Different liquidity costs $L(\rho) = |\rho|^{1+\phi} + \psi|\rho|$.

Bibliography

[1] Olivier GUÉANT, *The Financial Mathematics of Market Liquidity*, 2016.

Bibliography

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