Coherent Measures of Financial Risk

the Importance of Thinking Coherently

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Introduction: What is Coherence and Risk?

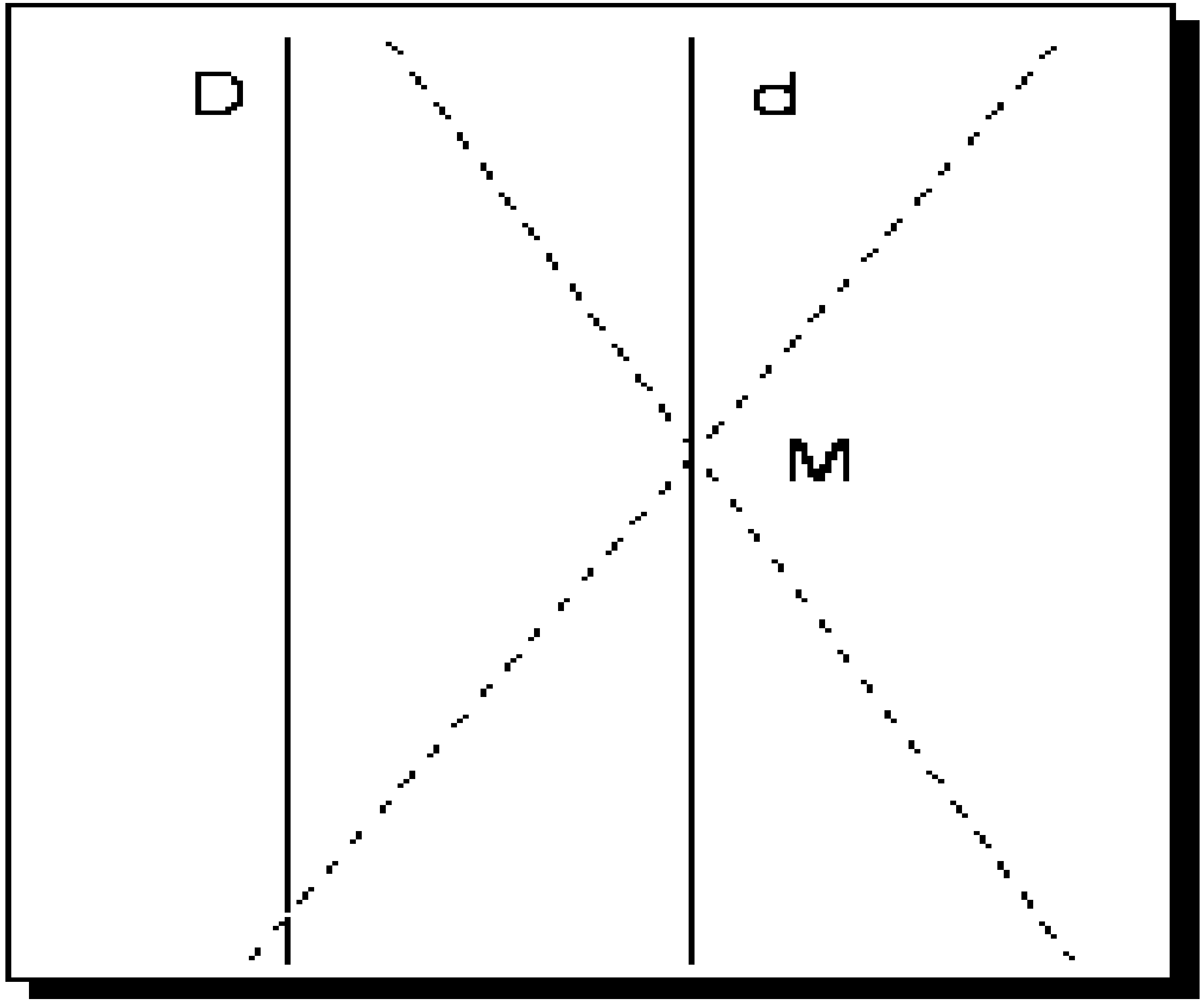

Figure 1: Euclid Proposed 5 Axioms (or rather 3 + 2 definitions) in his “Elements” as foundation of Geometry. — see eg. (Heath 1909)

Alternative Coherent Geometries

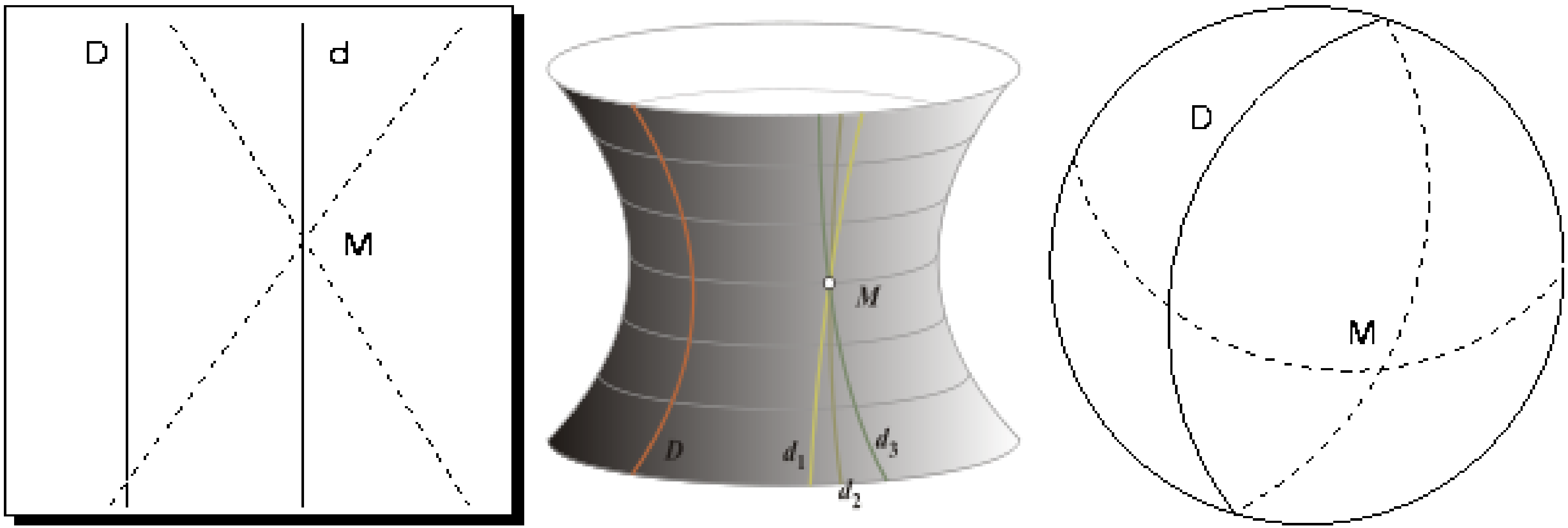

Figure 2: Alternative coherent geometries. Where in Ecuclid’s geometry there is exactly one line parallel to line D and through point M, in Nikolaï Lobatchevski’s hypersphere there are an infinite number and in Bernhard Riemann’s sphere there are none.

Thinking about Financial Risk

Idea	Reference
no risk, no rewards	Ecclesiastes 11:1–6 (ca. 300 BCE)
diversify investment	Ecclesiastes 11:1–2 (ca. 300 BCE) and Bernoulli (1738)

Table 1: Key ideas about investment risk

Thinking about Financial Risk

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Table 1: Key ideas about investment risk

Risk Measure	Reference
variance (**VAR**)	Fisher (1906), Marschak (1938) and Markowitz (1952)
Value at Risk (**V@R**)	Roy (1952)
semi-variance (**S**)	Markowitz (1991)
Expected Shortfall (**ES**)	Acerbi and Tasche (2002) and De Brouwer (2012)

Table 2: Normative theories and their risk measures implied.

Definitions of Risk Measures I

Definition 1 (Standard Deviation / Variance)

$$VAR := \textbf{variance} = E[(X - E[X])^2]$$
$$\sigma := \text{standard deviation} = \sqrt{VAR}$$

Definition 2 (Value-at-Risk)

$$V@R_\alpha(\mathcal{P}) := -(\text{the best of the } 100\alpha\% \text{ worst outcomes of } \mathcal{P})$$

Definitions of Risk Measures II

Definition 3 (Expected Shortfall)

$$ES_{(\alpha)}(\mathcal{P}) := -(\text{average of the worst } 100\alpha\% \text{ realizations})$$

Definition 4 (Worst Expected Loss)

$$WEL := \textbf{Worst Expected Loss} = -E[\min(\mathcal{P})]$$

Visualization of some risk measures

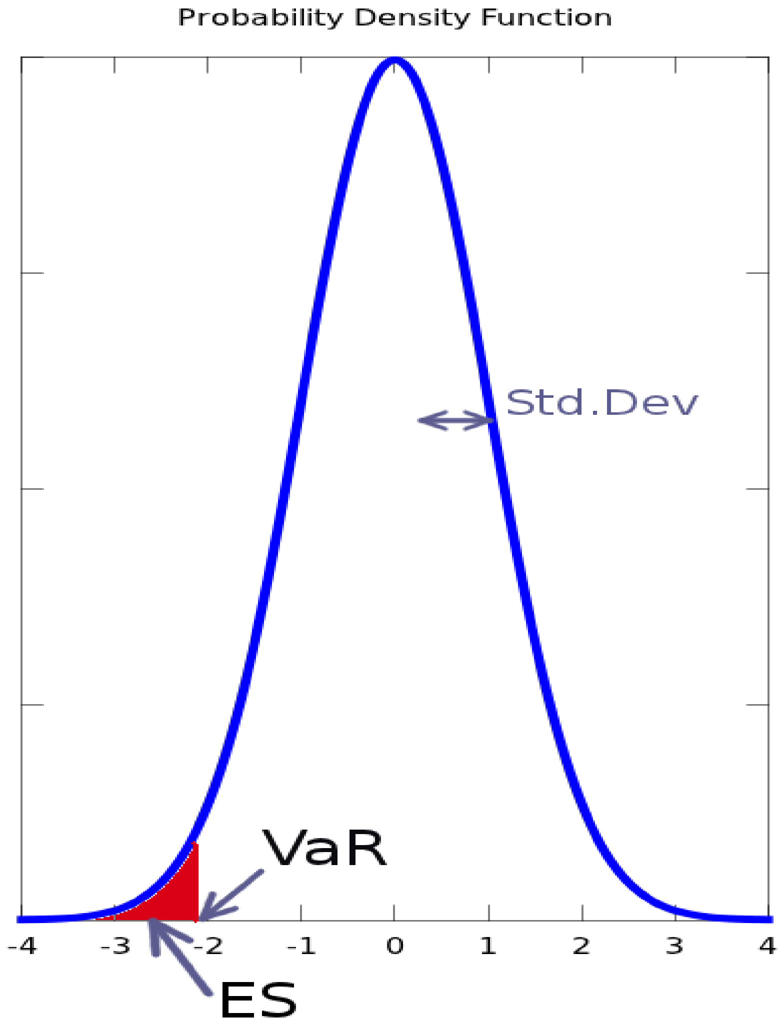
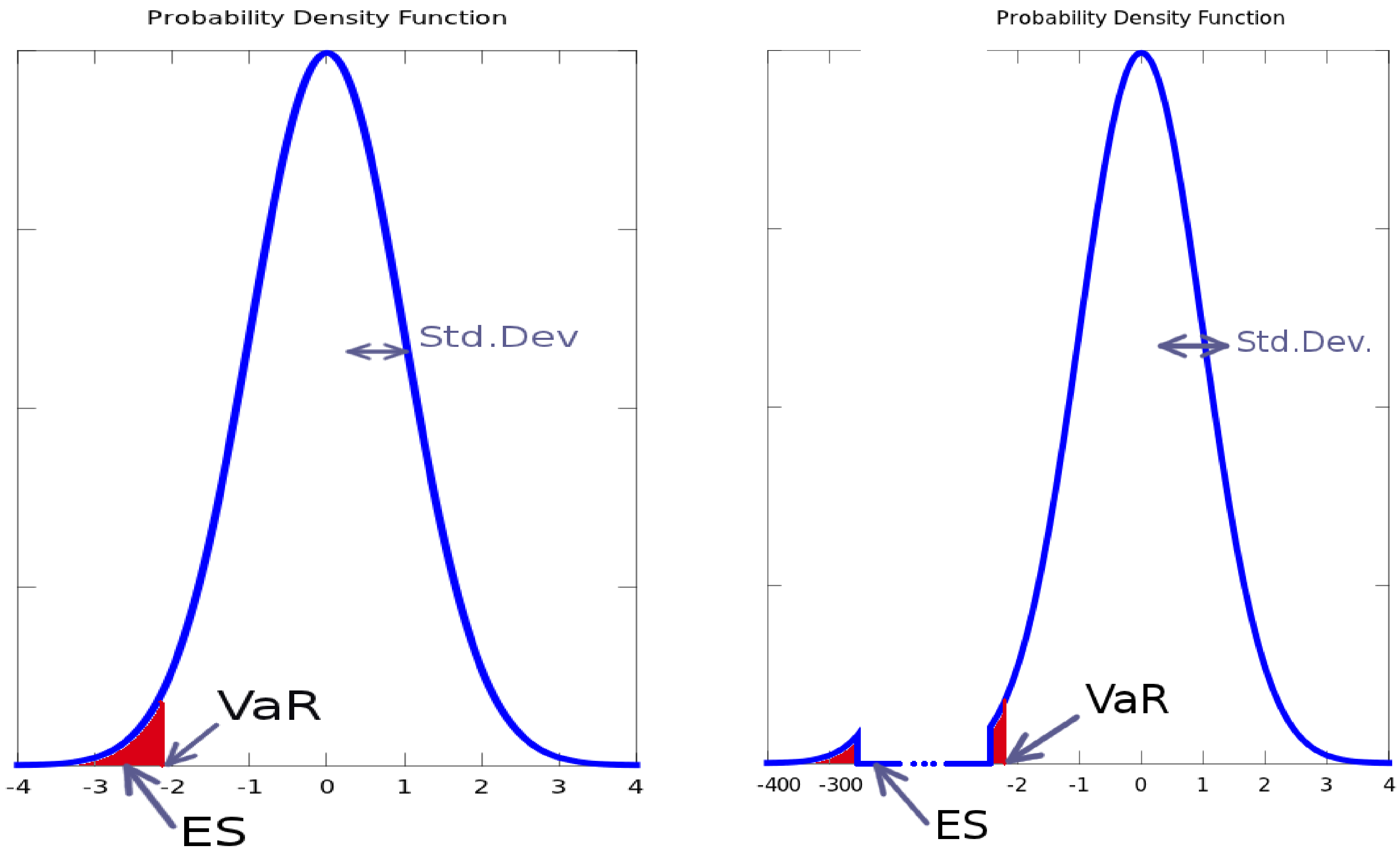


Figure 3: visualization of ES, V@R and σ. Note that WEL is not defined.

Visualization of some risk measures



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An Axiomatic Approach to Financial Risk

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A set of Axioms

Proposed by Artzner, Delbaen, Eber, and Heath (1997)

Definition 5 (Coherent Risk Measure)

A function $\rho : \mathbb{V} \mapsto \mathbb{R}$ is called a **coherent risk measure** if and only if

1. **monotonous**: $\forall X, Y \in \mathbb{V} : X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$

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4. **translation invariant**: $\forall a > 0$ and $\forall X \in \mathbb{V} : \rho(X + a) = \rho(X) - a$

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Law-invariance under P:

$\forall X, Y \in \mathbb{V}$ *and* $\forall t \in \mathbb{R} : P[X \leq t] = P[Y \leq t] \Rightarrow \rho(X) = \rho(Y)$

Which Risk Measure is Coherent?

- **VAR (or volatility)** is not coherent because it is not monotonous (trivial)
- **V@R** is not coherent, because it is not sub-additive (Artzner, Delbaen, Eber, and Heath 1999)
- **ES** is coherent (Pflug 2000)
- **WEL** is not usable because it is not Law-Invariant

. . . but who should care?

Definition 5 (Spectral Risk Measure)

Let X be a stochastic variable, representing the return of a financial asset. Then we define the **spectral measure of risk** $M_\phi(X)$ with **spectrum (or risk aversion function)** $\phi(p) : [0, 1] \mapsto \mathbb{R}$ as:

$$M_\phi(X) := -\int_0^1 \phi(p) F_X^{\leftarrow}(p)\, \mathrm{d}p$$

Coherence for Spectral Risk Measures

Theorem

The risk measure $M_\phi(X)$ as defined above is coherent, if and only if

$$\begin{cases} \phi(p) \text{ is positive} \\ \phi(p) \text{ is not increasing} \\ \int_0^1 \phi(p)\,\mathrm{d}p = 1 \end{cases}$$

Proof.

See (Acerbi 2002) □

The Spectrum of ES

Example 1

The spectrum or risk aversion function for the α-Expected Shortfall (ES$_\alpha$) is

$$\phi_{ES_\alpha}(p) = \frac{1}{\alpha}\mathbf{1}_{[p\leq\alpha]} := \begin{cases} \frac{1}{\alpha} & \text{if } p \leq \alpha \\ 0 & \text{else} \end{cases} \tag{1}$$

The Spectrum of V@R

Example 2

The spectrum or risk aversion function for the α-V@R is the Dirac delta function:

$$\phi_{V@R_\alpha}(p) = \delta(p - \alpha) \tag{2}$$

Case Studies

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Example 3 (One Bond)

Assume one bond with a 0.7% probability to default in one year in all other cases it pays 105% in one year. What is the $1\% V@R$?

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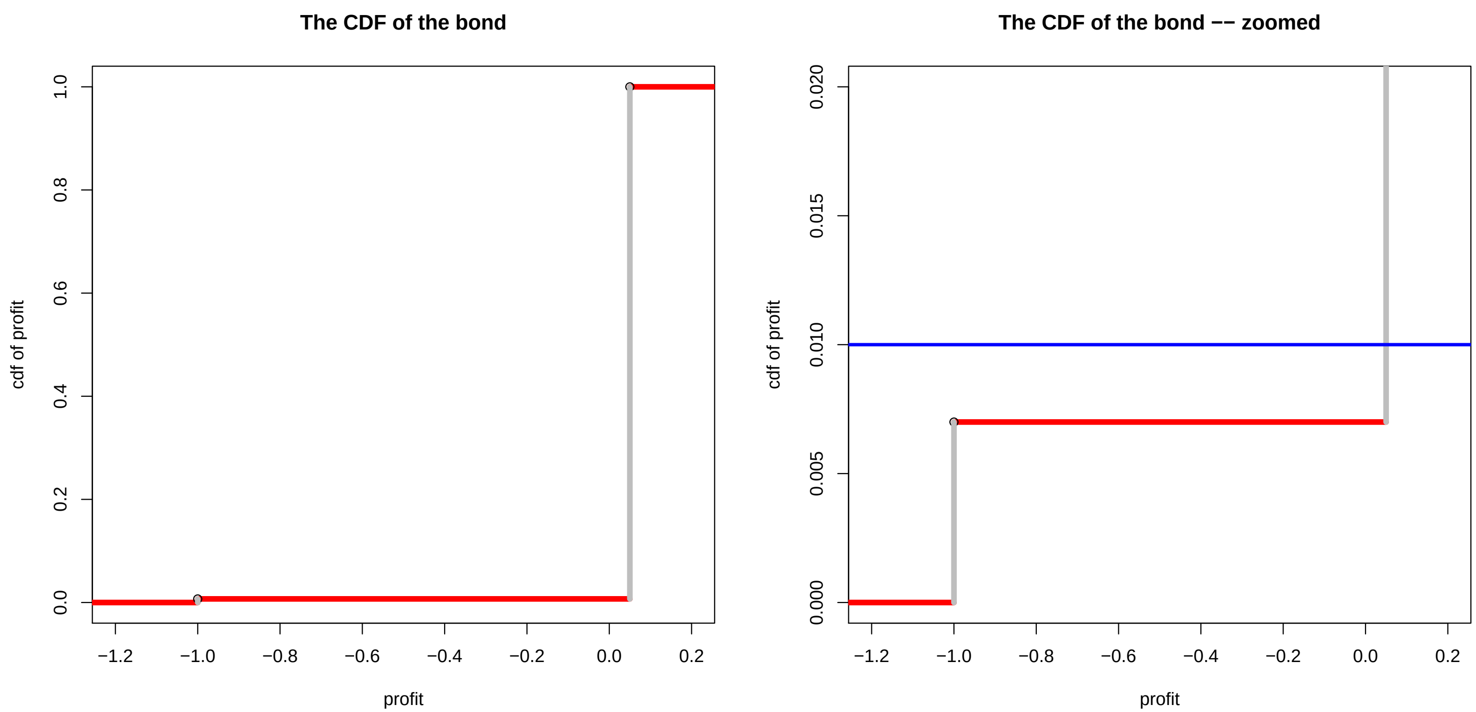

Case 2

Example 4 (Two Independent Bonds)

Consider two identical bonds with the same parameters, but independently distributed. What is the $1\% V@R$ now?

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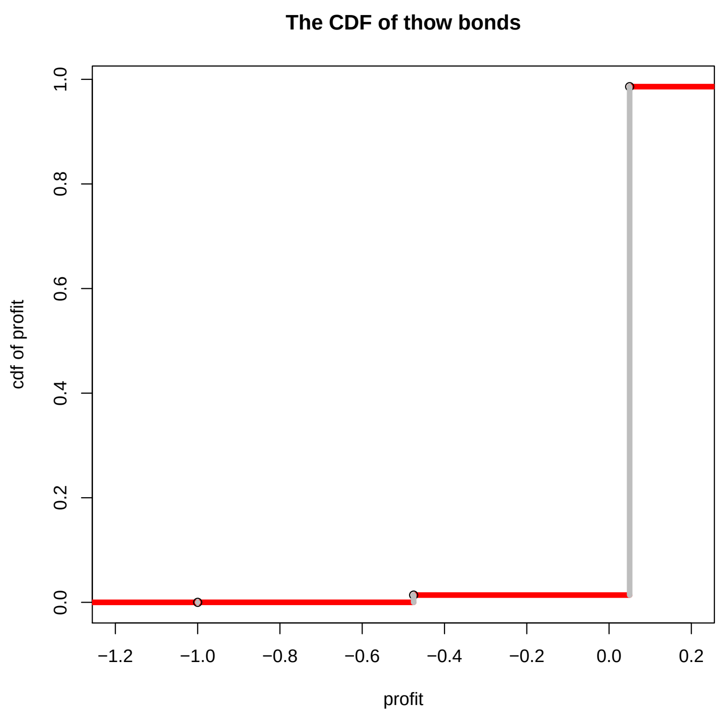

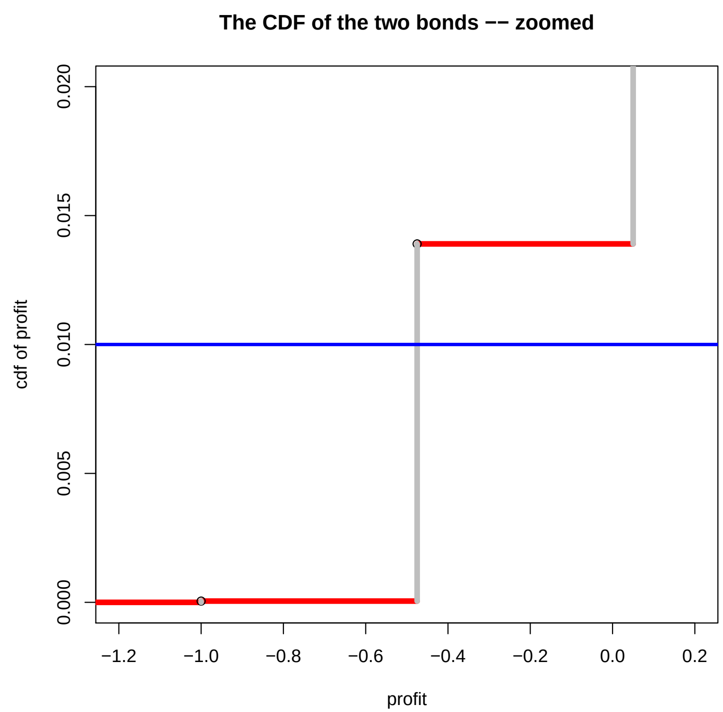

Case 3

The Evil Banker and his customers

Example 5 (The Evil Banker's First Dilemma)

Consider an Evil Banker who has to compose a portfolio for his private client. If there is at least one default in the portfolio, then the banker will loose that client.

How can our banker minimize his work and maximize his income?

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Case 4

The Evil Banker and Basel III

Example 6 (The Evil Banker's Second Dilemma)

Consider an Evil Banker who has to comply with Basel III, hence uses for assessing market risk $V@R$. Being Evil he does not care about the size of a bailout. So how does he minimize V@R?

Case 4
The Evil Banker and Basel III

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Case 5

More Bonds

Example 7 (N Independent Bonds)

Consider now an increasing number of independent bonds with the same parameters as in previous example.
Trace the risk surface.

Risk in Function of Diversification

Convecity (I)

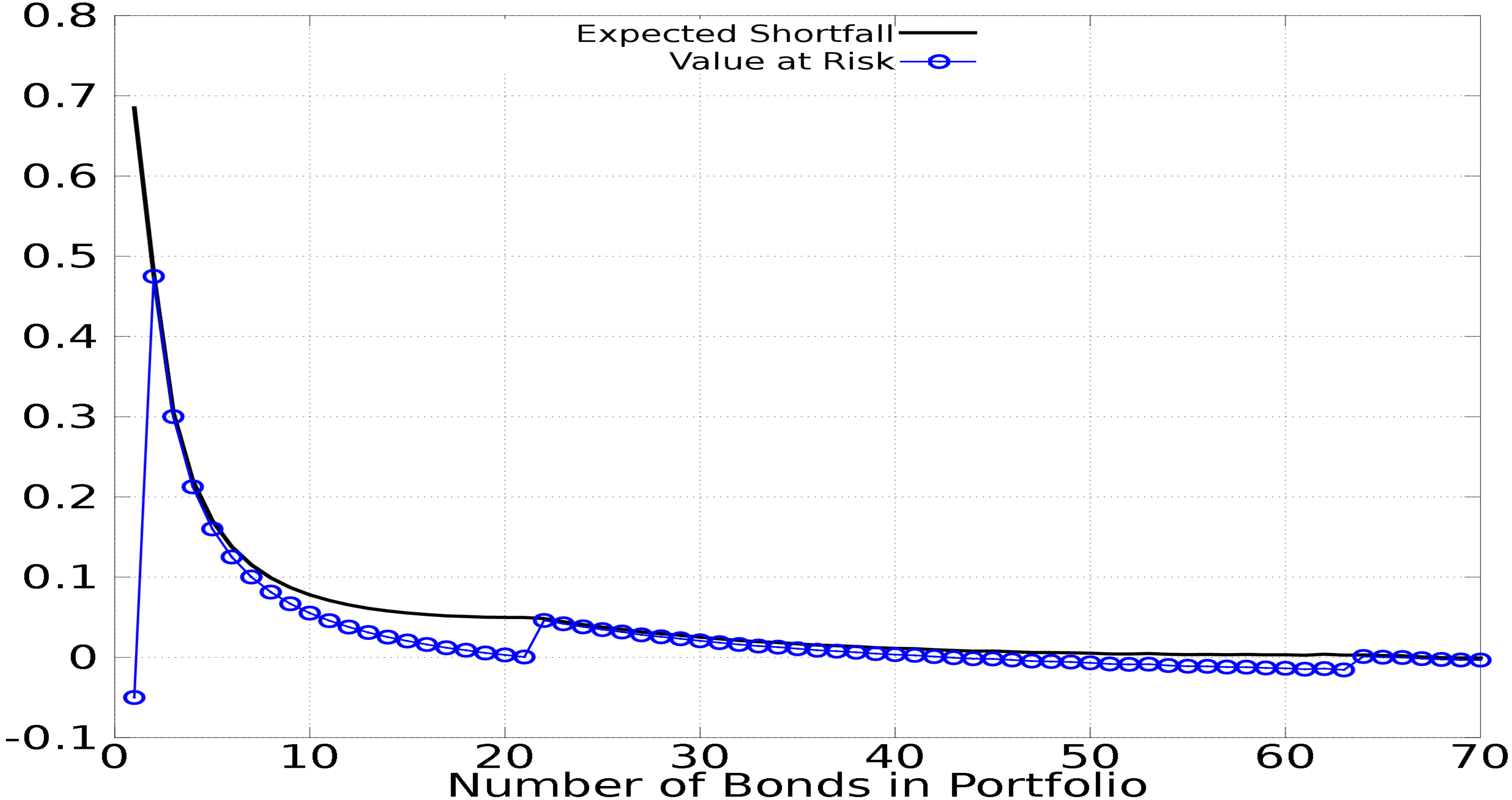

Figure 4: ES and V@R in function of number of bonds.

The Risk Surface

Convexity (II)

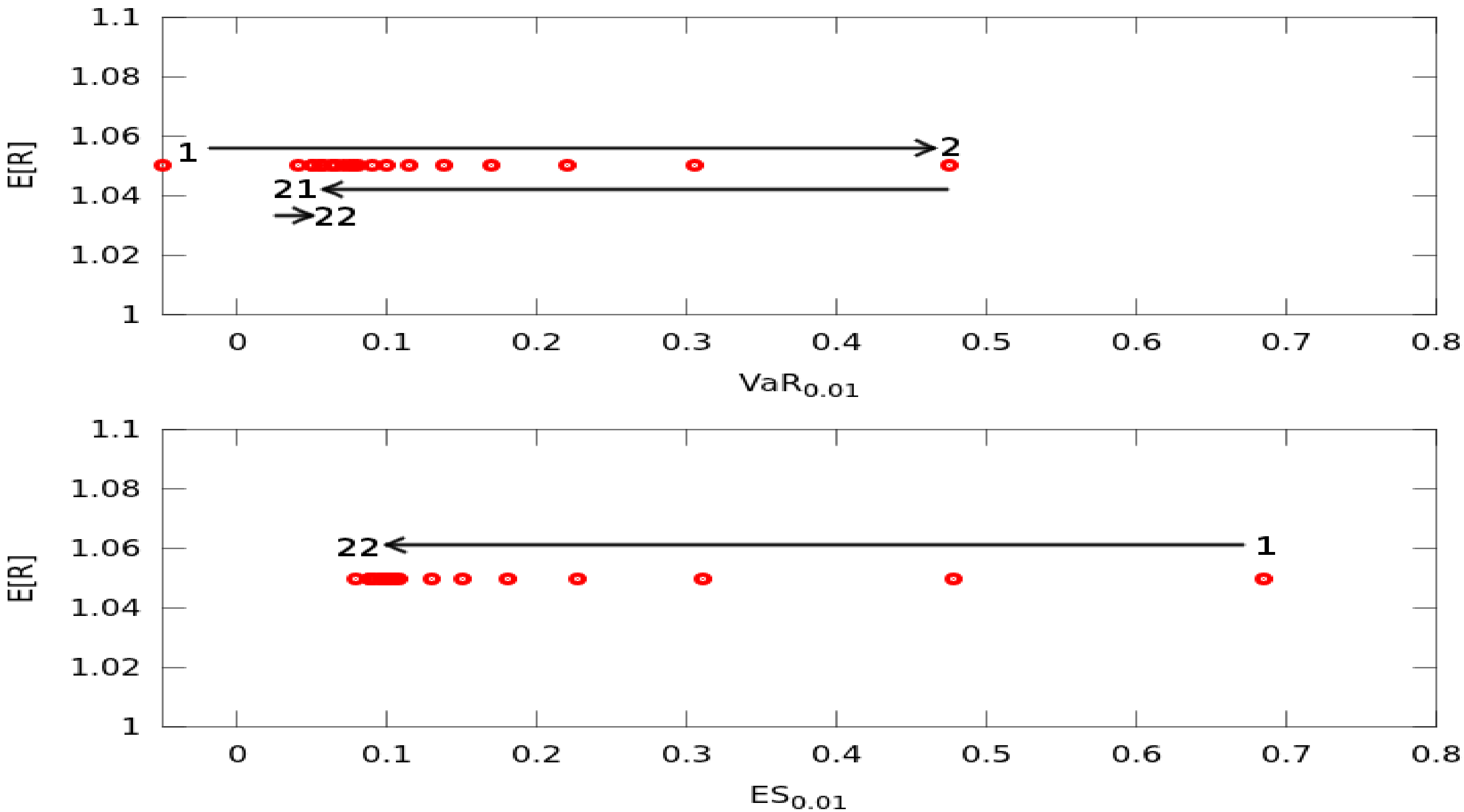

Figure 5: The result on the risk surface.

Case 6

Risk-Reward Optimization for Gaussian Returns

Example 8 (Three Gaussian Assets)

Consider three assets (or asset classes) that are all Gaussian (or at least elliptically) distributed and consider a risk-reward optimization

Optimal Portfolio Composition

The Mechanics of a Risk-Reward Method

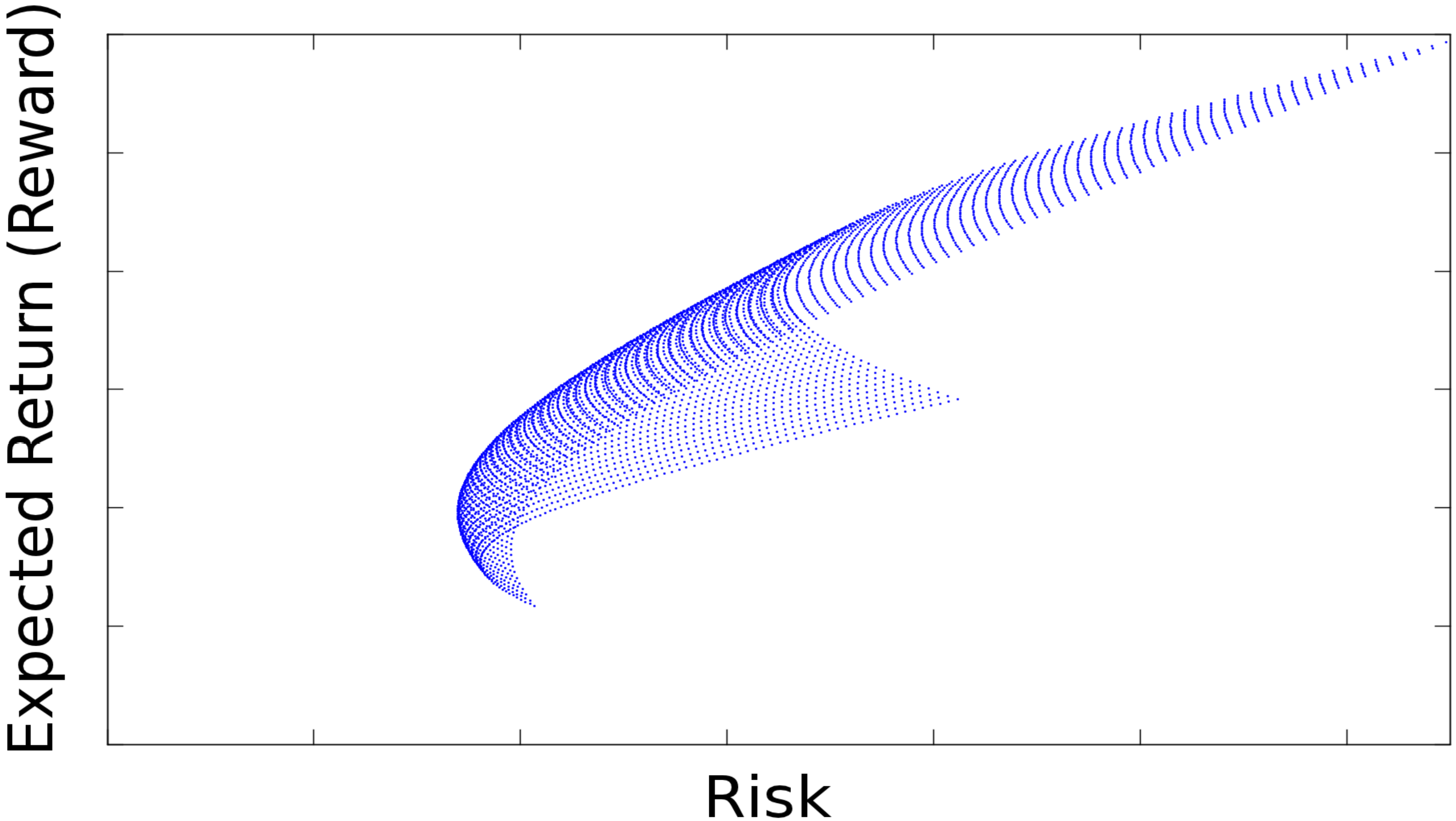

Figure 6: Portfolios in the risk/reward plane.

Example 1

Gausian Equities, Bonds and Cash—inflation adjusted

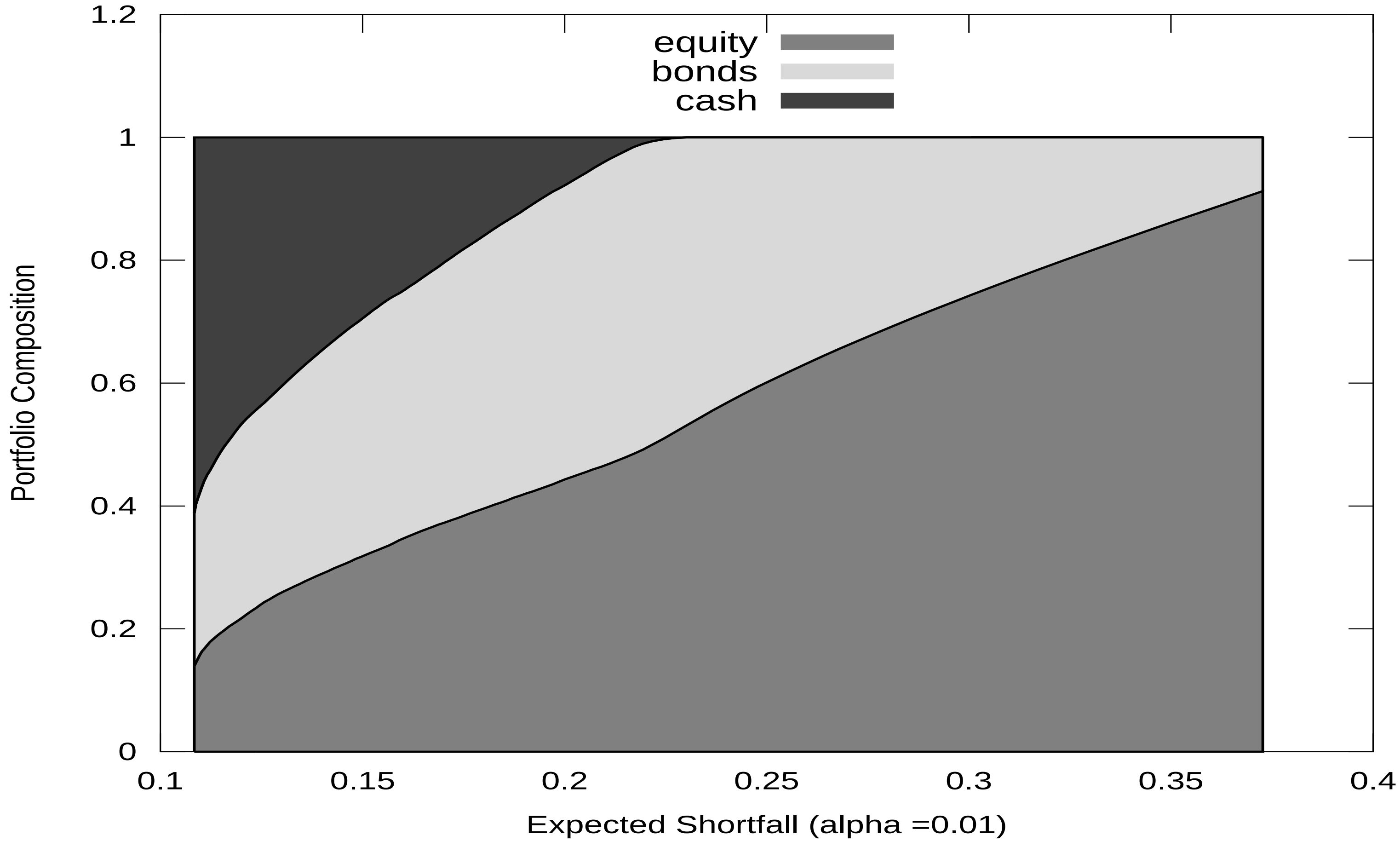

Figure 7: Recommended portfolios in function of ES.

Note that for Gaussian assets σ, *V@R* and *ES* lead to the same optimal portfolios.

Example 9 (Non-Gaussian Assets)

Consider three assets (or asset classes) that are all Gaussian distributed and consider a risk-reward optimization, but add a typical hedge fund and a typical capital guaranteed structure.

Case 7: Non-Gaussian Assets

The pdfs

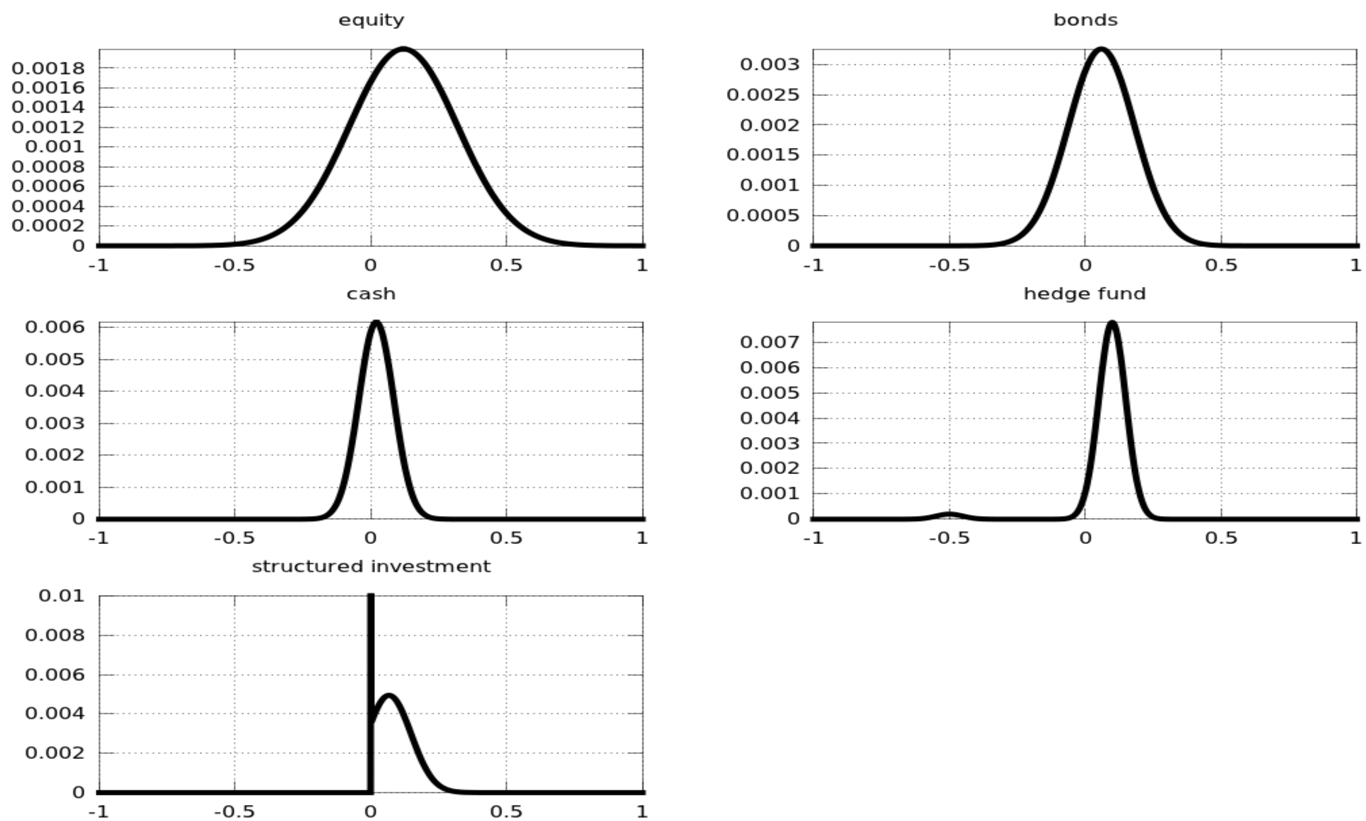

Figure 8: The pdfs in the example (the y-axis for the structured fund is truncated—this investment is a long call plus a deposit).

Case 7: Non-Gaussian Assets

Mean-ES and Mean-VAR Optimization

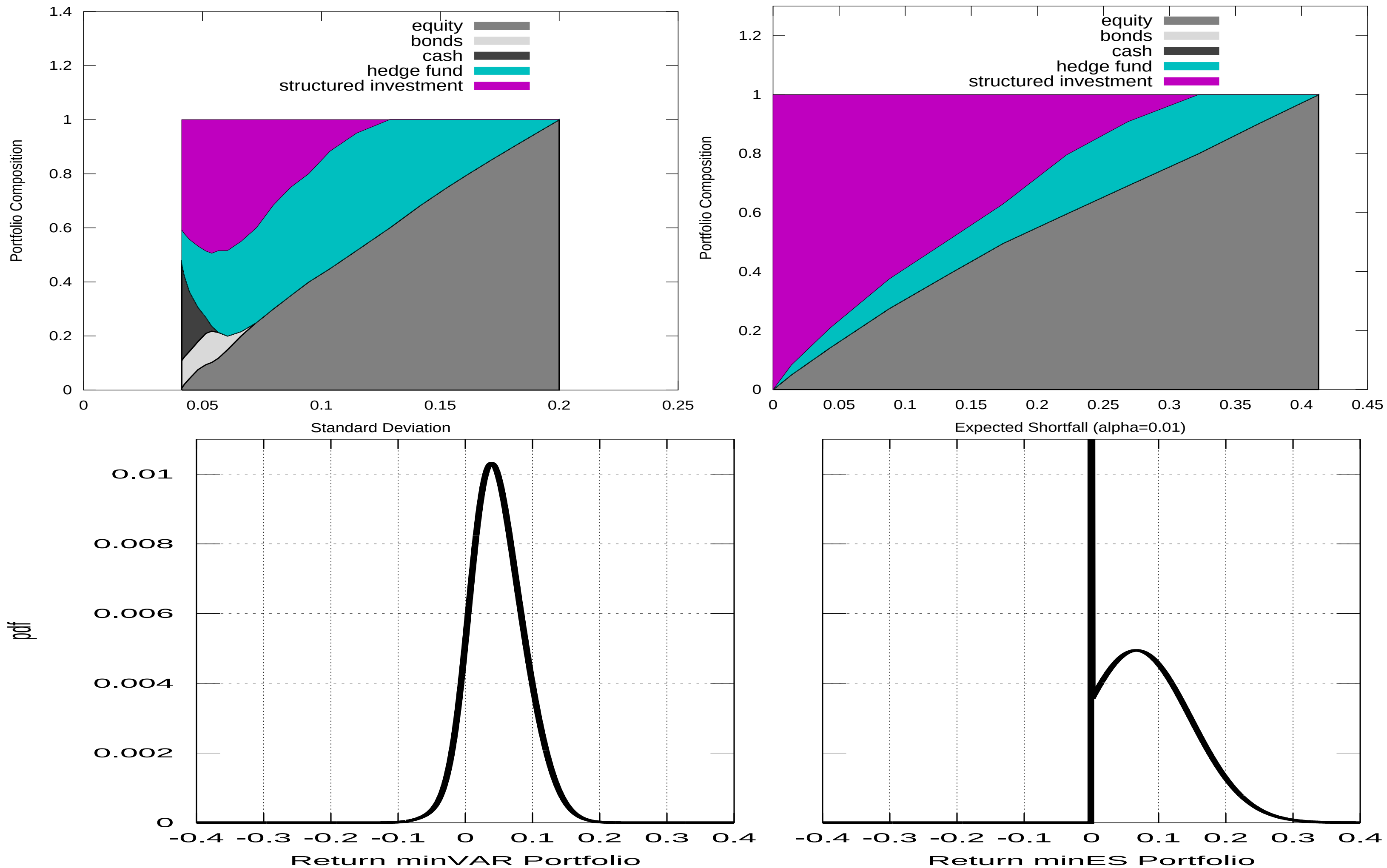

Figure 9: The min-VAR and min-ES portfolios compared.

Case 8: in UCITS IV legislation I

V@R as Risk Limit

For UCITS that are not managed relative to a benchmark UCITS IV defines the "Absolute V@R" limit:

$$V@R_{UCITS} \leq 20\% NAV$$

Example 10 (Risky Bet Fund)

Consider a structured fund that will pay in one year time 105% of the initial investment (assume that it pays the capital back plus a coupon of 5% in one year), except if company X defaults in that year, then it pays 0%. We estimate the probability that company X defaults in one year to equal 0.7%.
The $V@R_{UCITS}$ is -5%, so this is perfectly acceptable according to the General Guidelines of CESR/10-788.

Case 8: in UCITS IV legislation II

V@R as Risk Limit

Example 11 (Better Diversified Fund)

Consider a structured fund that will pay on one year time 105% of the initial investment, if either company X or Y defaults then it pays 52.5% of the initial investment, and if both companies X and Y default then it pays zero. We estimate the default probability of both company X and Y to equal 0.7%, and their default possibility is independently distributed.
The $V@R_{UCITS}$ is 47.5%, so this is not acceptable according to the General Guidelines of CESR/10-788.

Note: the same holds for the V@R limit in Basel II ICAAP. Examples: Lehman Brothers, Dexia, . . .

Case 9

A Risk Reward Indicator Based on Volatility (UICTS IV)

UCITS IV defines the “Risk Reward Indicator” as follows.

risk class	volatility equal or above	volatility less than
1	0%	0.5%
2	0.5%	2.0%
3	2.0%	5.0%
4	5.0%	10.0%
5	10.0%	15.0%
6	15.0%	25.0%
7	25.0%	$+\infty$

Table 3: The “risk classes” as defined by CESR in CESR/10-673, pg. 7, in the same document the risk classes are *also* referred to as “risk and reward indicator”.

Case 9

A Risk Reward Indicator Based on Volatility (UICTS IV)

Example 12 (Risk Classification)

Assume the assets from Example about non-Gaussian assets–example 7–plus one “risky bond” (this could also be a structured fund based on a digital option) that has a probability of 1% to loose 15% and a probability of 99% to gain 5%. Then consider the risk class as defined by CESR/10-673. The results are as in Table 4.

Case 9

A Risk Reward Indicator Based on Volatility (UICTS IV)

portfolio	risk class	σ	$ES_{0.01}$
equity	6	0.2000	0.4123
bonds	5	0.1200	0.2660
hedge fund	5	0.1062	0.5482
structured investment	4	0.0671	0.0000
risky bond	2	0.0198	0.1500
mix 1/2 equity + 1/2 bonds	5	0.1173	0.2223

Table 4: The risk classes for Example 36. CESR/ESMA's method considers the hedge fund that has roughly a 2.5% probability of loosing about 50% of its value is in the same risk class as a bond fund. A structured fund that has no risk to lose something ends up in the fourth risk class, but the risky bond that has a 1% probability of loosing 15% is considered as very safe!

Case 9

Bonus Example

Example 13 (The Evil Banker’s Third Dilemma)

How to reduce the risk class of the “risky bond” structure?

Case 10

Incoherence between the V@R-limit and the VAR-risk-class

risk limit, based on V@R

$$\Longleftrightarrow$$

risk classification, based on standard deviation

Example 14

Consider a structured fund that offers a 1% probability to loose 21% and a 99% probability to gain 5%. Such fund would not be possible, because its 1% $V@R_{UCITS}$ would be 21% (exceeding the limit and being classified as “too risky”). Its volatility is 2.5870%, that is only risk class 3, hence considered as safer than bonds—from our example, in the middle of the spectrum, and perfectly acceptable.

Incoherent risk measures in legislation

legislation	"risk measure"	result
UCITS	V@R and VAR	non suitable assets
Basel	V@R	crisis
Solvency	V@R	insolvency

Table 5: Law makers increasingly use non-coherent risk measures in legislation, resulting in encouraging to take large bets, ignore extreme risks and mislead investors. All building up to the next crisis . . . building up to the next global disaster.

Yes, it is important!

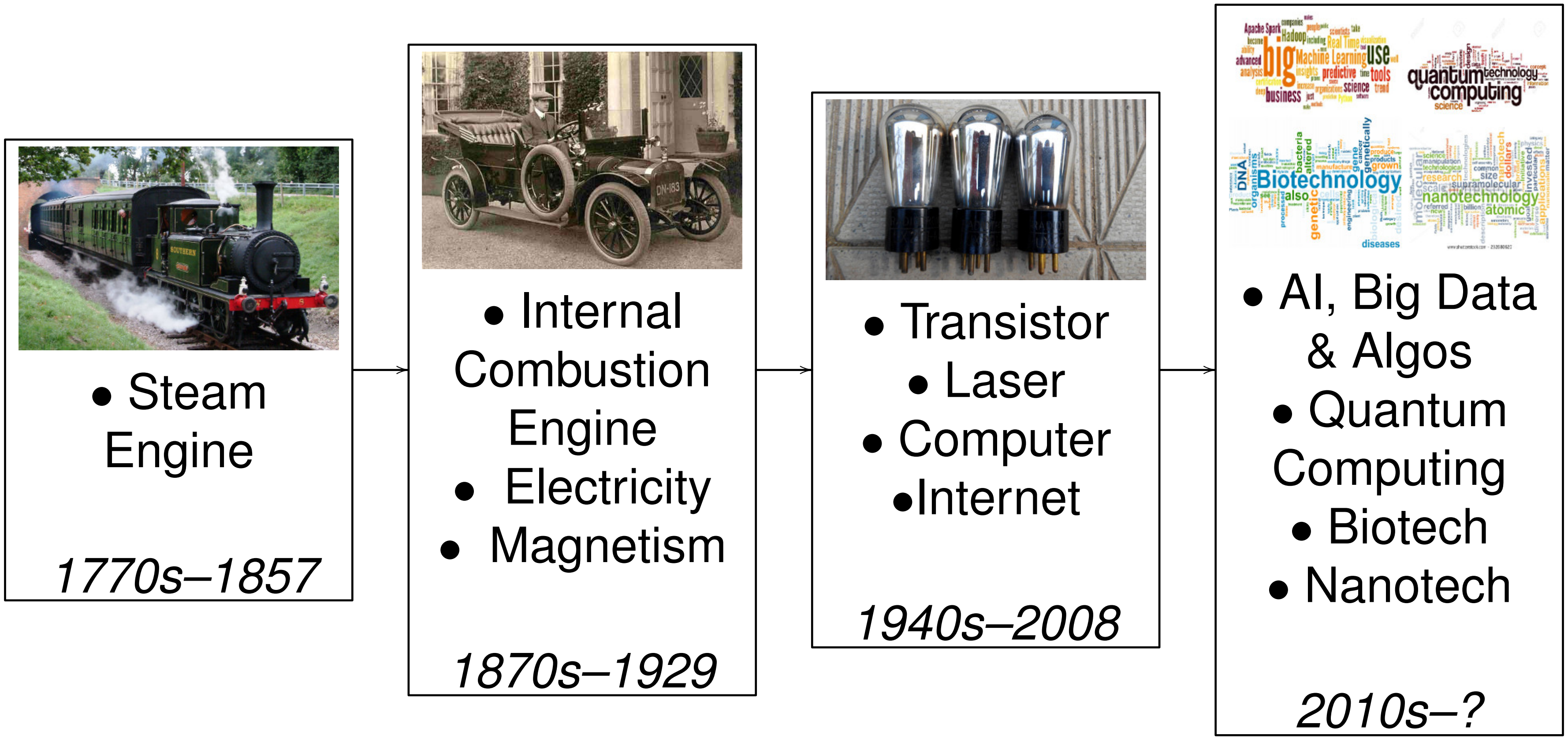

Figure 10: A simplified model of science propelling welfare and economy, but leading to crisis situations.

The Limits of Coherent Risk Measures

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The Limits of Coherent Risk Measures

Liquidity

The Limits of Coherent Risk Measures

Not a Real Valued Stochastic Variable

Example 15 (Basel II with ES?)

Would it make sense to replace $V@R$ in the capital requirements for banks by ES?

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Example 16 (Risk and Reward Indicator?)

Could a coherent risk measure be a “risk and reward indicator”?

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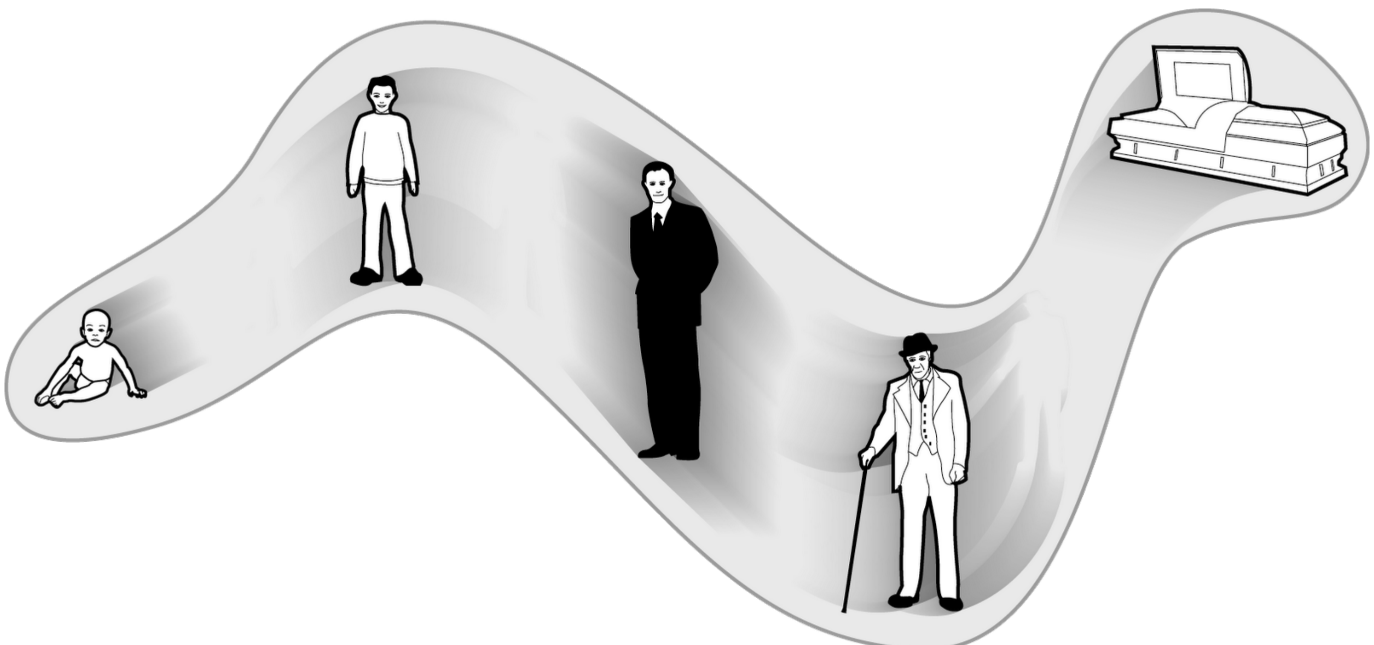

Conclusions

Formal Conclusions

Coherence does matter and its importance cannot be underestimated

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3. Hence, it is worth to make a rough estimate about the left tail of the distribution rather than ignoring it.

Formal Conclusions

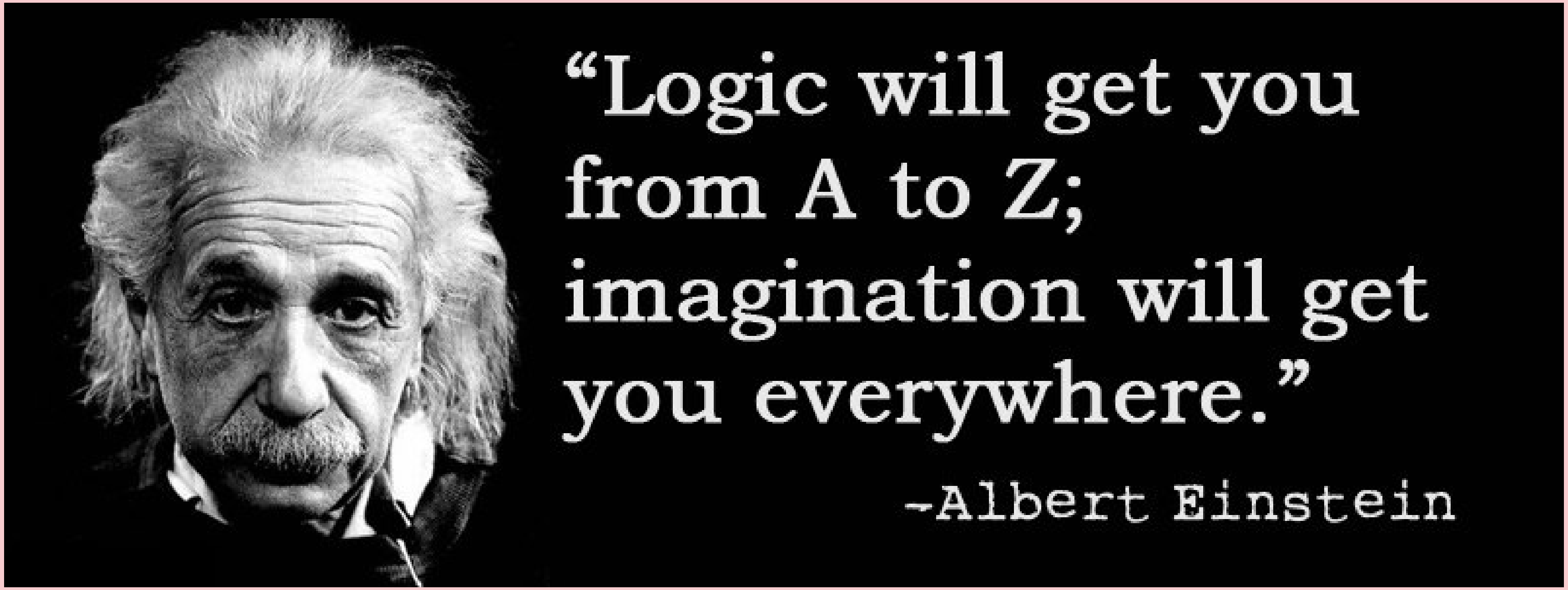
Coherence does matter and its importance cannot be underestimated

1. Coherence does matter.
2. An incoherent risk measure will lead to counter-intuitive and dangerous results.
3. Hence, it is worth to make a rough estimate about the left tail of the distribution rather than ignoring it.
4. Also Coherent Risk measures are a simplified reduction of the complex reality

Informal Conclusions

1. Let’s refer V@R to where it belongs: the museum of well-intended mistakes
2. **YOU** have a responsibility in this world!

THANK YOU FOR YOUR ATTENTION!



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